

18. Prove that given three distinct points z_1, z_2, z_3 in C_∞ , there exists a unique Mobius transformation $T(z)$ such that $T(z_1)=0$, $T(z_2)=1$ and $T(z_3)=\infty$.

19. State and prove Schwarz Lemma.

20. State and prove Riemann's Removable Singularity Theorem.

Time : Three hours

Maximum : 75 marks

SECTION A — (10 × 2 = 20 marks)

Answer ALL questions.

1. Define simply connected.
2. State Antiderivative Theorem.
3. Define Reparametrization.
4. Evaluate $\int_{\gamma} x dz$ where γ is the circle $|z|=R$.
5. Write the rotation property of Mobius maps.
6. What are the fixed points of $f(z)=z^2$?
7. Find the maximum value of $f(z)=e^{z-2}$ for $z \in \bar{\Delta}(a;r)$.
8. State Hadamard's Three Lines Theorem.

9. Define Removable singularity.

10. State Picard's Great Theorem.

SECTION B — (5 × 5 = 25 marks)

Answer ALL questions.

11. (a) Find the harmonic conjugate of $u(x, y) = 4xy - x^3 + 3xy^2$.

Or

(b) State and prove Uniqueness Theorem for power series.

12. (a) State and prove Weak form of Cauchy's Theorem.

Or

(b) Prove that for every closed contour γ in C and $a \in C \setminus \gamma$, $n(\gamma; a)$ is an integer.

13. (a) Let $f \in H(\Omega)$ and $z_0 \in \Omega$ such that $f'(z_0) \neq 0$. Prove that f is continuous at z_0 .

Or

(b) Find the image of the open upper semi-unit disk $\Omega = \{z : |z| < 1, \text{Im } z > 0\}$ under the Mobius map defined by $w = \varphi_a(z) = \frac{a-z}{1-\bar{a}z}$.

14. (a) State and prove Hadamard's Three Circles Theorem.

Or

(b) State and prove Liouville's Theorem.

15. (a) Let N be a deleted neighbourhood of z_0 such that f is analytic in N . If f has an isolated singularity at z_0 , show that z_0 is a pole of order n if and only if there are positive constants C_1 and C_2 such that $C_2 \leq |(z - z_0)^n f(z)| \leq C_1$ for some deleted neighbourhood N_0 of z_0 such that $N_0 \subset N$.

Or

(b) State and prove Picard's Little Theorem.

SECTION C — (3 × 10 = 30 marks)

Answer any THREE questions.

16. State and prove Ratio Test.

17. State and prove Cauchy - Goursat Theorem.